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On a problem of Rolewicz about Banach spaces that admit support sets

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Abstract

We construct an example of a nonseparable Banach space which does not admit a support set.[2] It is a consistent (and necessarily independent from the axioms of ZFC) example of a space $C(K)$ of continuous functions on a compact Hausdorff K with the supremum norm. The construction depends on a construction of a Boolean algebra with some combinatorial properties. The space is also hereditarily Lindelöf in the weak topology but it doesn't have any nonseparable subspace nor any nonseparable quotient which is a $C(K)$ space for K dispersed.

Keywords: Banach space; Biorthogonal system; Banach spaces of continuous functions; Independence results

1. Introduction

A nonempty closed convex set C in a Banach space X is called a *support set* if and only if for every point $z \in C$ there is a continuous linear functional ϕ on X such that $\phi(z) = \inf_C \phi <$

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[2] Similar results (but using quite a different counterexample) were announced roughly at the same time by S. Todorcevic.

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$\sup_C \phi$. In [13], Rolewicz proved that assumptions needed for solving a certain natural problem can be weakened in the case of a separable Banach space. Namely that there are no separable support sets. Rolewicz also proved in [13] that in a nonseparable Hilbert space there are convex sets which are support sets and he asked if every nonseparable Banach space has a support set.

This problem has been addressed in several papers where extracting certain uncountable sequences in a Banach space is used to construct (necessarily nonseparable) support set. In [8] a Marcuševič's basis is used for this purpose, in [9] properties of a compact Hausdorff K like being not hereditarily separable or not hereditarily Lindelöf are employed in the corresponding $C(K)$ and in [2] an equivalent condition in terms of sequences from $X \times X^*$ is found. In [4], besides more detailed analysis in Banach lattices, it is proved that if K carries a nonseparable regular measure, then $C(K)$ has a support set and the same conclusion is obtained for some other class of Banach spaces in [10].

In this paper we provide an example of a nonseparable Banach space with no support set. It is a consistent example of a nonseparable $C(K)$ space i.e., a Banach space of continuous functions on a Hausdorff compact space K with the supremum norm. As S. Todorcevic has proved in [17] that it is consistent that any nonseparable Banach space contains an uncountable biorthogonal system, it follows that the nonexistence of our example is also consistent with the axioms of ZFC, leaving Rolewicz's problem undecidable. Todorcevic's result for $C(K)$ for K totally disconnected follows from Boolean algebraic results of [16].

Let us describe the context of the paper in more details.

In Section 2 we describe the properties of the Boolean algebra of clopen sets of our K (K will be totally disconnected, and hence the Stone space of a Boolean algebra i.e., a particular case of the structure space with Gelfand's topology for the abelian C^*-algebra $C(K)$). K is in a sense obtained from the Cantor set by splitting just once each point of an uncountable set $\{r_\xi\colon \xi < \omega_1\}$ of points of the Cantor set. Already a classical example of the split interval (i.e., $[0,1] \times \{0,1\}$ with the lexicographical order topology) is of this form, however as proved in [3, Example II 5.1], the corresponding $C(K)$ has a support set. There are of course other ways of splitting single points r_ξ than in the split interval, namely using two open $U_1, U_2 \subseteq 2^\omega$ with $U_1 \cup U_2 = 2^\omega - \{r_\xi\}$ and $\{r_\xi\} = \overline{U}_1 \cap \overline{U}_2$. This can be applied to the resulting spaces and can be done in many ways, for example like in [14] or [15], [1] or in [6] where first countable compact spaces with continuous images with no first countable points are obtained, being the biggest in this context.

Our present construction is somewhere in between the construction of [14] where the splitting of r_ξ's is done completely independently of other splittings and n-Rubin algebras constructed in [15] and compact spaces constructed in [1] where some repeated splitting can happen unlike in our case. A rigorous comparison of the structure of such spaces can be done using certain trees considered for example in [6]. It can be seen that $C(K)$ obtained from the Boolean algebras of [14] have uncountable biorthogonal systems for the same reason as for the split interval. On the other hand the complexity of the compact spaces obtained from the algebras of [15] and [1], as it seems to the author, makes the analysis of their Radon measures a much more challenging task. Actually it is the example from [1] that is claimed in Remark 5 of [17] also not to have a support set.

In our case there are simple combinatorial properties, stated in Definition 1 and Theorem 7 of the Boolean algebra and its Stone space needed for the proofs of all the properties of the corresponding $C(K)$. Also in this section we obtain a representation of simple functions (Lemma 4) on K which is used to prove that our space $C(K)$ is hereditarily Lindelöf in the weak topology (Theorem 8). The only example of such a nonseparable Banach space $C(K)$ which appears in the literature is Kunen's space (see [11]), however it is an Asplund space (equivalently K is dispersed) and ours $C(K)$ is not. The characterization of Banach spaces which admit a support set of [2] easily implies that subspaces and quotients of a space which does not admit a support set also do not admit it. On the other hand it follows from results of [9] that every nonseparable $C(K)$ for K dispersed admits a support set. It follows that our $C(K)$ is quite far from an Asplund $C(K)$s and in particular from Kunen's example.

In Section 3 we prove the consistency of the existence of the algebra. It is done using the method of forcing (see [7]), i.e., we consider a partial order of finite approximations to the final algebra. The approximations can be amalgamated to obtain the relations as in Theorem 7. This section can be skipped by a reader not familiar with forcing. The following section refers just to the statements of the results of Section 2.

The forcing method guarantees that it is consistent that there is a direct limit $\mathcal{A}$ of such finite approximations such that in any uncountable family of finite subalgebras of $\mathcal{A}$ two of them are amalgamated in $\mathcal{A}$ in such a way that the relations of Theorem 7 are satisfied. This is a technically simple way of showing the consistency, however it is heuristically complex due to the use of the forcing method. Another way of the construction, with reversed advantages and disadvantages, would probably be to use the $\diamondsuit$ principle of Jensen, for example like in [15]. The proof does not go through assuming just the continuum hypothesis (recall that $\diamondsuit$ is stronger than CH, see [7]) for example along the lines of [14].

In Section 4 we embark on analyzing right-separated sequences in the dual ball of the $C(K)$ with the weak* topology, where K is the Stone space of the algebra obtained in the previous sections. Recall that a sequence $(x_\alpha\colon\ \alpha < \omega_1)$ of points of a topological space is called right-separated (left-separated) if and only if $x_\beta \notin \overline{\{x_\alpha\colon\ \alpha > \beta\}}$ ($x_\beta \notin \overline{\{x_\alpha\colon\ \alpha < \beta\}}$) for every $\beta < \omega_1$. It is well known that a regular topological space has an uncountable right-separated (left-separated) sequence if and only if it is not hereditarily Lindelöf (not hereditarily separable) (see e.g. [12, Theorem 3.1]) i.e., all subspaces (equivalently all closed subspaces) are Lindelöf (separable). Note that for $(x_\alpha\colon\ \alpha < \omega_1)$ left-separated means that there are open (basic if needed) U_α's for $\alpha < \omega_1$ such that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ for $\beta < \alpha$. On the other hand for $(x_\alpha\colon\ \alpha < \omega_1)$ to be right-separated means that there are open (basic if needed) U_α's for $\alpha < \omega_1$ such that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ for $\beta > \alpha$.

Note that, if K is compact and nonmetrizable, and hence without a countable base, applying the Hahn–Banach theorem one can choose a sequence $(f_\alpha, \mu_\alpha)_{\alpha<\omega_1} \subseteq C(K) \times C(K)^*$ such that $\mu_\alpha(f_\alpha) = 1$ and $\mu_\beta(f_\alpha) = 0$ for all $\alpha < \beta < \omega_1$ i.e., the dual ball of a nonseparable $C(K)$ cannot be hereditarily Lindelöf in the weak* topology (this applies to any nonseparable Banach space). We show that in any such sequence (in our $C(K)$), there are $\alpha < \beta < \omega_1$ such that $\mu_\alpha(f_\beta) < 0$. Recall a definition of Borwein and Vanderwerff from [2]. Let X be a Banach space. A sequence $(x_\alpha, \phi_\alpha)_{\alpha<\kappa}$ of elements of $X \times X^*$ is called semibiorthogonal if and only if

(i) $\phi_\alpha(x_\alpha) = 1$,
(ii) $\phi_\beta(x_\alpha) = 0$ if $\alpha < \beta$,
(iii) $\phi_\alpha(x_\beta) \geqslant 0$ if $\alpha < \beta$.

Hence, we show that our $C(K)$ has no uncountable semibiorthogonal sequences. As it is proved in [2] that X has a support set if and only if it has a semibiorthogonal sequence of length ω_1, we conclude that $C(K)$ has no support sets, providing a negative answer to Rolewicz's question.

The notation and terminology is quite standard. We often identify the dual to a $C(K)$ with the space $M(K)$ of Radon measures on K with the variation norm.

2. A Boolean algebra, its Stone space and its continuous functions

Let us fix the terminology concerning Boolean algebras and the Stone duality. The complement of $a = 1a$ in a Boolean algebra is denoted by $-a$ or $-1a$, the supremum, the infimum, the least element and the biggest element are denoted $\vee$, $\wedge$, 0, 1, respectively. The basic clopen sets of the Stone space of a Boolean algebra $\mathcal{A}$ consisting of all ultrafilters of $\mathcal{A}$ containing a are denoted by a^*. We have $(a \wedge b)^* = a^* \cap b^*$, etc. We will consider the Boolean order where $a \leqslant b$ if and only if $a \wedge b = a$. Recall that any subset of the algebra which has the finite intersection property (i.e., every finite subset has infimum different than 0) extends to an ultrafilter.

A family $(U_i\colon i \in \omega)$ included in a Boolean algebra $\mathcal{A}$ is called independent if and only if

$$\epsilon_0 U_0 \wedge \cdots \wedge \epsilon_n U_n \neq 0$$

for all $\epsilon_0, \dots, \epsilon_n \in \{-1, 1\}$ and every $n \in \omega$. By U_s we denote $\epsilon_0 U_0 \wedge \cdots \wedge \epsilon_n U_n$ where $s \in \{-1, 1\}^{n+1}$ is such a finite function $s : \{0, \dots, n\} \to \{-1, 1\}$ that $s(i) = \epsilon_i$ for all $i \leqslant n$. Since any element of the Boolean algebra generated by an independent family as above is the supremum of a finite family of elements of the form U_s, the ultrafilters of the algebra are generated by the sets $\{U_s\colon x|n = s\}$ for all $x \in \{-1, 1\}^\omega$, and the topology of the Stone space is the usual topology of the Cantor set identified with $\{-1, 1\}^\omega$ with the product topology.

Consider another example of the Stone space of a Boolean algebra included in the power set of $\{-1, 1\}^\omega \times \{-1, 1\}$ generated by the sets of the form

$$\begin{aligned} a_r &= \bigl\{(x, i)\colon x < r,\ i = -1, 1\bigr\} \cup \bigl\{(r, -1)\bigr\}, \\ -a_r &= \bigl\{(x, i)\colon x > r,\ i = -1, 1\bigr\} \cup \bigl\{(r, 1)\bigr\}, \\ U_s &= \bigl\{(x, i)\colon x|n = s,\ i = -1, 1\bigr\}, \end{aligned}$$

where $r \in \{-1, 1\}^\omega$, $<$ is the lexicographical order and $s \in \{1, -1\}^n$ for some $n \in \omega$. Any ultrafilter of this Boolean algebra intersected with the algebra generated by the independent family must be an ultrafilter of the latter i.e., of the form $\{U_s\colon x|n = s\}$ for some $x \in \{-1, 1\}^\omega$. Now note for any $r \neq x$, there is an n such that either $U_{r|n} \subseteq a_r$ or $U_{r|n} \subseteq -a_r$. Since ultrafilters are closed under bigger elements of the algebra, we conclude that all ultrafilters of the algebra are generated by the sets of the form $\{U_s\colon x|n = s\} \cup \{a_x\}$ or $\{U_s\colon x|n = s\} \cup \{-a_x\}$. One can check that the Stone topology is exactly of "the split Cantor set" i.e., the subspace of the split interval corresponding to the Cantor set.

Our example will be obtained by splitting, in a more complicated way, some (arbitrary) uncountable sequence $\{r_\xi\colon \xi < \omega_1\}$ of distinct elements of the Cantor set $\{-1, 1\}^\omega$. However, the Stone space of the algebra can be interpreted as a subspace of $\{-1, 1\}^\omega \times \{-1, 1\}$ with some compact Hausdorff topology. The following definition lists the properties of the generators and the following lemma concludes the form of the ultrafilters of the algebra i.e., the form of the neighbourhood bases of the points.

Definition 1. Suppose that $(r_\xi\colon \xi < \omega_1)$ is a sequence of distinct elements of $\{-1, 1\}^\omega$. Let $\mathcal{A}$ be a Boolean algebra generated by an independent family $(U_i\colon i \in N)$ and additional generators $(a_\xi\colon \xi < \omega_1)$ which satisfy:

(a) For every $\xi < \omega_1$ and $\epsilon \in \{-1, 1\}$ the family $\{U_{r_\xi|i}\colon i \in N\} \cup \{\epsilon a_\xi\}$ has the finite intersection property.
(b) If $\delta \in \{-1, 1\}$ and $\eta < \xi < \omega_1$, then there is $i \in N$ such that

$$U_{r_\eta|i} \wedge \delta a_\eta \leqslant a_\xi \quad \text{or} \quad U_{r_\eta|i} \wedge \delta a_\eta \leqslant -a_\xi.$$

(c) If $r \in \{-1, 1\}^\omega - \{r_\eta\colon \eta \leqslant \xi\}$, then there is $i \in N$ such that

$$U_{r|i} \leqslant a_\xi \quad \text{or} \quad U_{r|i} \leqslant -a_\xi.$$

Then the Stone space of $\mathcal{A}$ is called an unordered split Cantor set. For such a space a_ξ, r_ξ, U_i will refer to the objects as above.

Remark. (a) says that a_ξ splits r_ξ for every $\xi < \omega_1$; (b) says that if a_ξ splits r_η for $\eta < \xi$, then it does it, locally, the same way as a_η; (c) says that a_ξ does not split r_η's for $\eta > \xi$ nor the points of $\{-1, 1\}^\omega$ outside of the sequence $\{r_\eta\colon \eta < \omega_1\}$.

Lemma 2. *Suppose that $\mathcal{A}$ is the algebra whose Stone space is an unordered split Cantor set. Then, all ultrafilters of $\mathcal{A}$, that is the points of the Stone space, are of the following two forms*:

(1) *generated by $(U_{x|n}\colon n \in N)$ if and only if $x \in (\{-1, 1\}^\omega - \{r_\xi\colon \xi < \omega_1\})$, which will be denoted by $[x]$,*
(2) *generated by $(U_{r_\xi|n}\colon n \in N) \cup \{\epsilon a_\xi\}$ for $\xi < \omega_1$ and any $\epsilon \in \{-1, 1\}$, which are denoted by $[r_\xi, 1]$, $[r_\xi, -1]$, respectively.*

The set $\{[r_\xi, 1], [r_\xi, -1]\}$ will be denoted by R_ξ.

Proof. Prove the above properties by induction on $0 \leqslant \alpha \leqslant \omega_1$ for algebras A_α generated by the sequence $(U_i\colon i \in N)$ and generators $(a_\xi\colon \xi < \alpha)$ satisfying (a)–(c) of Definition 1 for $\eta, \xi < \alpha$ instead of $\eta, \xi < \omega_1$. The limit step, in particular for $\alpha = \omega_1$ is trivial since if a set generates an ultrafilter in algebras belonging to an increasing chain, it generates an ultrafilter in the algebra which is the union of the chain.

To make the successor step, note that an ultrafilter in a bigger algebra $A_{\beta+1}$ intersected with the subalgebra A_β is an ultrafilter in the subalgebra. If this intersection is like in (1) or in (2) for $\xi \neq \beta$, the conditions (b), (c) of Definition 1 imply that the ultrafilter (of $A_{\beta+1}$) is already generated from A_β. Otherwise (a) implies that both of the possible extensions are ultrafilters. □

Lemma 3. *Suppose that $\mathcal{A}$ is a clopen algebra of an unordered split Cantor set. Let $\mathcal{A}_\alpha$ denote the subalgebra of $\mathcal{A}$ generated by $(U_i\colon i \in \omega)$ and $\{a_\xi\colon \xi < \alpha\}$ for $\alpha \leqslant \omega_1$. For every $n \in \omega$ we have*

$$a_\alpha - U_{r_\alpha|n} \in \mathcal{A}_\alpha.$$

Proof. Let K denote the Stone space of $\mathcal{A}$. Any point of $K - R_\alpha$ has a neighbourhood a^* included in a_α^* or disjoint from a_α^* for $a \in \mathcal{A}_\alpha$ by Definition 1 (b) and (c). Since $a_\alpha^* - U_{r_\alpha|n}^*$ is a compact subspace of this set, we have a finite subcover consisting of subsets i.e., $a_\alpha - U_{r_\alpha|n}$ is the supremum of a finite family of elements of $\mathcal{A}_\alpha$ as required. □

Let $\mathcal{A}_\alpha$ be as in the previous lemma and let K_α be its Stone space. It is well known that there is a canonical isometric embedding of $C(K_\alpha)$ into $C(K_{\omega_1})$. We will identify the image of this embedding with $C(K_\alpha)$, in particular for $\alpha = 0$ we have a copy of $C(\{-1,1\}^\omega)$ inside $C(K_{\omega_1})$.

Let us see the general form of continuous rational simple functions on unordered split interval. By a rational simple function we mean a function assuming only finitely many rational values.

Lemma 4. *Suppose that K is an unordered split Cantor set, $\varepsilon > 0$, μ is a (regular) Radon measure on K and that f is a continuous rational simple function on K. Then there is a simple rational function $g \in C(\{-1,1\}^\omega)$, distinct $\xi_1, \dots, \xi_k < \omega_1$ and there are rationals q_i, non-negative integers m_i and $s_i \in \{-1,1\}^{m_i}$ such that $s_i = r_{\xi_i}|m_i$ for $1 \leqslant i \leqslant k \in \omega$ such that*

$$f = g + \sum_{1 \leqslant i \leqslant k} q_i \chi_{a^*_{\xi_i} \cap U^*_{s_i}}$$

and such that

$$\sum_{1 \leqslant i \leqslant k} |q_i||\mu|\big(U^*_{s_i} - R_{\xi_i}\big) \leqslant \varepsilon,$$

where $R_\xi = \{[r_\xi, 1], [r_\xi, -1]\}$.

Proof. Consider subalgebras $\mathcal{A}_\xi$ of $\mathcal{A}$ for $\xi \leqslant \omega_1$ generated by U_i's and a_η's for $\eta < \xi$ and their Stone spaces K_ξ. By induction on ξ we prove that any continuous simple rational functions in $C(K_\xi)$ can be written in the form as in the lemma.

It is clear that the limit stage is trivial. So suppose we are done for A_ξ and we are given a continuous simple rational function f on $K_{\xi+1}$. Note that

$$\bigcap_{m \in N} U^*_{r_\xi|m} = R_\xi$$

since, by Lemma 2, the only ultrafilters in $\mathcal{A}_{\xi+1}$ which extend the filter generated by $\{U_{r_\xi|n}: n \in N\}$ are $\{[r_\xi, 1], [r_\xi, -1]\}$. Hence, the regularity of the Radon measures, implies that $|\mu|(U^*_{r_\xi|n} - R_\xi)$ converge to 0. Let n_1 be such that

$$|\mu|\big(U^*_{r_\xi|n} - R_\xi\big) \leqslant \frac{\varepsilon}{4\|f\|}$$

for $n \geqslant n_1$.

Note also, that a simple function is a linear combination of characteristic functions of clopen sets, i.e., sets corresponding to elements of $\mathcal{A}$, hence there is a finite set $\xi_1, \dots, \xi_{k-1} < \xi < \omega_1$ such that preimages under f belong to the subalgebra of $\mathcal{A}_{\xi+1}$ generated by U_i's for $i < n_2$ and $a_{\xi_1}, \dots, a_{\xi_{k-1}}, a_\xi$. Now let $n \geqslant n_1, n_2$ be such that for every $1 \leqslant i < k$ either $U_{r_\xi|n} \leqslant a_{\xi_i}$ or $U_{r_\xi|n} \leqslant -a_{\xi_i}$ for $1 \leqslant i \leqslant k-1$ which can be obtained by Definition 1(c).

It follows that f is constant on $a^*_\xi \cap U^*_{r_\xi|n}$ and is constant on $U^*_{r_\xi|n} - a^*_\xi$. Let q_1, q_2 be the corresponding values, note that $|q_1 - q_2| \leqslant 2\|f\|$. So

$$f = \big[f|\big(K_\xi - U^*_{r_\xi|n}\big) + q_2\chi_{U^*_{r_\xi|n}}\big] + (q_1 - q_2)\chi_{a^*_\xi \cap U^*_{r_\xi|n}}.$$

Note that $f|(K_\xi - U^*_{r_\xi|n})$ belongs to $C(K_\xi)$ by Lemma 3, and so

$$f = h + q_k \chi_{a^*_\xi \cap U^*_{r_\xi|n}}, \qquad |q_k||\mu|\big(U^*_{r_\xi|n} - R_\xi\big) \leqslant \frac{\varepsilon}{2}$$

where $q_k = q_1 - q_2$ and $h \in C(K_\xi)$. Hence the inductive assumption for $\varepsilon/2$ can be used which completes the proof of the lemma. □

Lemma 5. *Suppose $n \in N$ and μ^l_α for $1 \leqslant l \leqslant n$ and $\alpha < \omega_1$ are nonatomic Radon measures on an unordered split Cantor set K. Suppose U_α for $\alpha < \omega_1$ are open sets in $M(K)^n$ where $M(K)$ is considered with the weak* topology and suppose that for all $\alpha < \omega_1$ we have*

$$\big(\mu^1_\alpha, \dots, \mu^n_\alpha\big) \in U_\alpha.$$

Then there exists an uncountable set $A \subseteq \omega_1$ such that for every $\alpha, \beta \in A$ we have $(\mu^1_\beta, \dots, \mu^n_\beta) \in U_\alpha$.

Proof. We may w.l.o.g. assume that $U_\alpha = U_{\alpha,1} \times \dots \times U_{\alpha,n}$ where

$$U_{\alpha,l} = \bigcap_{1 \leqslant j \leqslant p_l} \big\{\mu\colon \big(\mu - \mu^l_\alpha\big)(h_{\alpha,l,j}) < \varepsilon\big\}$$

where $h_{\alpha,l,j} \in C(K)$ for $1 \leqslant l \leqslant n$ and $1 \leqslant j \leqslant p_l$, $p_l \in N$ and $\varepsilon > 0$. We may also w.l.o.g. assume that the norms of all the measures μ^l_α are bounded by some $\delta > 0$. Let $f_{\alpha,l,j}$ be simple rational functions satisfying

$$\|f_{\alpha,l,j} - h_{\alpha,l,j}\| < \varepsilon/6\delta.$$

By the previous lemma and the fact that there are countably many rational simple functions in $C(\{-1,1\}^\omega)$ as well as there are countably many clopen subsets of $\{-1,1\}^\omega$ and countably many rationals, there is an uncountable set $A' \subseteq \omega_1$ and there exist $g_{l,j} \in C(\{-1,1\}^\omega)$, distinct $\xi^{\alpha,l,j}_1, \dots, \xi^{\alpha,l,j}_{k_{l,j}} < \omega_1$ and there are rationals $q_{l,j,i}$, non-negative integers $m_{l,j,i}$ and $k_{l,j}$ and sequences $s_{l,j,i} \in \{-1,1\}^{m_{l,j,i}}$ such that $s_{l,j,i} = r_{\xi^{\alpha,l,j}_i}|m_{l,j,i}$ for $1 \leqslant i \leqslant k_{l,j} \in \omega$ such that for every $\alpha \in A'$ we have

$$f_{\alpha,l,j} = g_{l,j} + \sum_{1 \leqslant i \leqslant k_{l,j}} q_{l,j,i} \chi_{a^*_{\xi^{\alpha,l,j}_i} \cap U^*_{s_{l,j,i}}}$$

and such that

$$\sum_{1 \leqslant i \leqslant k_{l,j}} |q_{l,j,i}|\big|\mu^l_\alpha\big|\big(U^*_{s_{l,j,i}} - R_{\xi^{\alpha,l,j}_i}\big) \leqslant \varepsilon/3,$$

where $R_{\xi^{\alpha,l,j}_i} = \{[r_{\xi^{\alpha,l,j}_i}, 1], [r_{\xi^{\alpha,l,j}_i}, -1]\}$. Since the measures μ^l_α are nonatomic and $R_{\xi^{\alpha,l,j}_i}$s consist of two points of K we obtain that

$$\sum_{1\leqslant i\leqslant k_{l,j}} |q_{l,j,i}|\left|\mu^l_\alpha\right|\left(U^*_{s_{l,j,i}}\right) \leqslant \varepsilon/3.$$

Moreover there exists an uncountable $A \subseteq A'$ such that for every $\alpha, \beta \in A$ and every $1 \leqslant j \leqslant p_l$ for $1 \leqslant l \leqslant n$ we have

$$\left|\left(\mu^l_\beta - \mu^l_\alpha\right)(g_{l,j})\right| < \varepsilon/3.$$

Hence for $\alpha, \beta \in A$

$$\begin{aligned}\left|\left(\mu^l_\beta - \mu^l_\alpha\right)(h_{\alpha,l,j})\right| &\leqslant \left|\left(\mu^l_\beta - \mu^l_\alpha\right)(f_{\alpha,l,j})\right| + 2\delta\cdot\varepsilon/6\delta \\ &\leqslant \left|\left(\mu^l_\beta - \mu^l_\alpha\right)(g_{l,j})\right| + \varepsilon/3 + \varepsilon/3 < \varepsilon\end{aligned}$$

which means that $\mu^l_\beta \in U_{\alpha,l}$ which means that $(\mu^1_\beta, \ldots, \mu^n_\beta) \in U_\alpha$ which completes the proof. □

Proposition 6. *Suppose that K is an unordered split Cantor set. The dual of $C(K)$ (in the norm) is isomorphic to the dual of $C(\{-1,1\}^\omega)$. For every positive integer $n \in N$ the n-th power $M^n(K)$ of the dual of $C(K)$ in the weak* topology is hereditarily separable if and only if for every $n \in N$ the space $l^n_1(K)$ is hereditarily separable in the weak* topology.*

Proof. It is well known that for any K the dual $M(K)$ to $C(K)$ is isomorphic to $NA(K) \oplus l_1(|K|)$ where $NA(K)$ is the subspace of $M(K)$ consisting of nonatomic Radon measures. Since the cardinalities of K and $\{-1,1\}^\omega$ are the same we will obtain the required norm isomorphism between the dual spaces of $C(\{-1,1\}^\omega)$ and $C(K)$ if we can show that $NA(\{-1,1\}^\omega)$ is isomorphic to $NA(K)$. The isomorphism will just be the restriction.

First, we will show that if μ is a nonatomic Radon measure on K which is null on $\mathcal{A}_0$ i.e., the clopen sets generated by the independent family, then $\mu = 0$.

Indeed, fix μ as above and take any $f \in C(K)$ which is rational simple function and $\varepsilon > 0$. By Lemma 4 we have

$$f = g + \sum_{1\leqslant i\leqslant k} q_i \chi_{a^*_{\xi_i} \cap U^*_{s_i}}$$

with

$$\sum_{1\leqslant i\leqslant k} |q_i||\mu|\left(U^*_{s_i} - R_{\xi_i}\right) \leqslant \varepsilon,$$

where all the objects are like in Lemma 4. As μ is null on $\mathcal{A}_0$, we have $\int g\, d\mu = 0$. The fact that μ is nonatomic implies that $\mu(R_{\xi_i})$ are all zero. So, the above inequality implies that $|\int f\, d\mu| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary we have that $\int f\, d\mu = 0$, for any simple rational function f. But such functions are norm dense in $C(K)$ by the Stone–Weierstrass theorem, so $\mu = 0$ as required.

By the above observation to obtain an isomorphism between the space of nonatomic measures on $C(\{-1,1\}^\omega)$ and the space of nonatomic measures of $C(K)$, we need to show that the restrictions of nonatomic measures on $C(K)$ to $C(\{-1,1\}^\omega)$ (treated as functionals) are nonatomic and that every nonatomic measure on $C(\{-1,1\}^\omega)$ has a nonatomic extension to $C(K)$. These follow from the fact that the points of $\{-1,1\}^\omega$ are split just in two points of K.

Now let us turn to the dual of $C(K)$ considered with the weak* topology. As $l_1(K)$ with the weak* topology is its subspace, to prove the second part of the proposition it is enough to prove that $M^n(K)$ is hereditarily separable for every $n \in N$ assuming that $l_1^n(K)$ has this property for every $n \in N$.

First see that every finite power of $NA(K) \times l_1(K)$ is hereditarily separable where $NA(K)$ is the space of nonatomic measures on K with the weak* topology. If not, there would be an uncountable left-separated sequence (for definition and relevance see the introduction) in some finite power of $NA(K) \times l_1(K)$. By Lemma 5, it would reduce to an uncountable left-separated sequence in a finite power of $l_1(K)$ contradicting the hereditary separability of that space (note that the product of two hereditarily separable spaces does not have to be hereditarily separable). To conclude that every finite power of the dual to $C(K)$ is hereditarily separable it is enough to note that $M(K)$ is a continuous image of $NA(K) \times l_1(K)$, addition being the mapping where all spaces are considered in the weak* topology. □

Theorem 7. *It is consistent that there is a Boolean algebra $\mathcal{A}$ whose Stone space is an unordered split Cantor set such that given any collection of pairwise disjoint sets $F_\alpha = \{\xi_\alpha^1, \ldots, \xi_\alpha^k\} \subseteq \omega_1$ for $\alpha < \omega_1$, $\epsilon_i \in \{-1,1\}$, $\delta_i \in \{-1,1\}$ for all $1 \leqslant i \leqslant k$ and $a \subseteq \{1,\ldots,k\}$ there are $\alpha < \beta$ such that*

$$R_{\xi_\beta^i} \subseteq a^*_{\xi_\alpha^i}, \quad \text{if } i \in a,$$

$$R_{\xi_\beta^i} \subseteq -a^*_{\xi_\alpha^i} \quad \text{if } i \notin a$$

for all $1 \leqslant i \leqslant k$, and there are $\alpha < \beta$ such that

$$[r_{\xi_\alpha^i}, 1] \in \epsilon_i a^*_{\xi_\beta^i},$$

$$[r_{\xi_\alpha^i}, -1] \in \delta_i a^*_{\xi_\beta^i}$$

for all $1 \leqslant i \leqslant k$. The Stone space of such an algebra will be called a generic unordered split Cantor set.

The proof of the above theorem is the subject of the following section. Now, however, we can prove some properties of a generic unordered split Cantor sets.

Theorem 8. *Suppose that K is a generic unordered split Cantor set. For every $n \in N$, the product K^n is hereditarily separable. K is hereditarily Lindelöf and K^2 is not. $C(K)$ is hereditarily Lindelöf in the weak topology and the dual space of $C(K)$ with the weak* topology is hereditarily separable.*

Proof. Note that to prove that K^n is hereditarily separable or that K is hereditarily Lindelöf it is sufficient (by thinning out the sequences) to work with points of K of the form $[r_\xi, \epsilon]$ since the remaining points form a subspace of the metrizable Cantor set.

So assume that $(x_1^\alpha, \dots, x_n^\alpha) \in K^n$ for $\alpha < \omega_1$ are of this form. Making an inductive argument one can assume without loss of generality (again thinning out the sequence) that $x_i^\alpha \neq x_j^\alpha$ if $i \neq j$. Now, consider some neighbourhoods of these points in K^n, they could be basic neighbourhoods, i.e., of the form $U_{s_i^\alpha} \cap \epsilon_i a_{\xi_i^\alpha}$ for distinct fixed s_i^α's in some 2^m. But the second part of Theorem 7 implies that there are $\alpha < \beta$ such that $x_i^\alpha \in \epsilon_i a^*_{\xi_i^\beta}$ i.e., the sequence is not left-separated.

To prove that K is hereditarily Lindelöf use the same argument and the first part of Theorem 7.

Finally to see that the square K^2 is not hereditarily Lindelöf. Consider the sequence $([r_\xi, 1], [r_\xi, -1])_{\xi<\omega_1}$ with the neighbourhoods $(a_\xi^* \times (-a_\xi^*))_{\xi<\omega_1}$. As a_η does not split the set R_ξ for $\xi > \eta$ by Definition 1(c), we have $\{[r_\xi, 1], [r_\xi, -1]\} \subseteq a_\eta^*$ or $\{[r_\xi, 1], [r_\xi, -1]\} \cap a_\eta^* = \emptyset$ if $\eta < \xi$. This implies that

$$\big([r_\xi, 1], [r_\xi, -1]\big) \notin a_\eta^* \times \big(-a_\eta^*\big)$$

for every $\eta < \xi$ i.e., K^2 has an uncountable right-separated sequence.

Now we turn to the properties of the Banach space $C(K)$ and its dual $M(K)$. Besides Proposition 6 we will need two folklore facts that every finite power of a Banach space X is hereditarily Lindelöf in the weak topology if and only if every finite power of X^* is hereditarily separable in the weak* topology and that every finite power of $l_1(K)$ is hereditarily separable if and only if every finite power of K is hereditarily separable. Both of them we will implicitly reproved below in the needed directions.

We will be using again the fact that a regular topological space X is not hereditarily separable if and only if it has an uncountable left-separated sequence and it is not hereditarily Lindelöf if and only if it has an uncountable right-separated sequence.

First we will prove that the dual is hereditarily separable in the weak* topology. The first step is to note that the fact that K^n is hereditarily separable for every $n \in N$ implies that for every $n \in N$ the space $l_1(K)^n$ is hereditarily separable in the weak* topology.

Suppose that $\{\mu_\alpha^l\colon \alpha < \omega_1,\ 1 \leqslant l \leqslant n\}$ is a collection of measures in $l_1(K)$ such that for $x_\alpha^* = (\mu_\alpha^1, \dots, \mu_\alpha^n)$ the sequence $(x_\alpha^*)_{\alpha<\omega_1}$ is left-separated i.e., as before there are open sets U_αs such that we have $x_\alpha^* \in U_\alpha$ and $x_\beta \notin U_\alpha$ for $\beta < \alpha$. Using this notation we may w.l.o.g. assume that $U_\alpha = U_{\alpha,1} \times \dots \times U_{\alpha,n}$ where

$$U_{\alpha,l} = \bigcap_{1 \leqslant j \leqslant p_l} \big\{\mu\colon \big|\big(\mu - \mu_\alpha^l\big)(h_{\alpha,l,j})\big| < \varepsilon\big\}$$

where $h_{\alpha,l,j} \in C(K)$ for $1 \leqslant l \leqslant n$ and $1 \leqslant j \leqslant p_l$, $p_l \in N$ and $\varepsilon > 0$ such that $x_\alpha^* \in U_\alpha$ and $x_\beta^* \notin U_\alpha$. We may assume without loss of generality that all $h_{\alpha,l,j}$'s are bounded by some $M > 0$. Now for each $\alpha < \omega_1$ and $1 \leqslant l \leqslant n$ find $\nu_\alpha^l \in l_1(K)$ such that

$$\big\|\nu_\alpha^l - \mu_\alpha^l\big\| < \varepsilon/3M$$

and ν_α^l is a finite rational linear combination of pointwise measures. I.e., without loss of generality we may assume that for all $\alpha < \omega_1$ there are rationals $q_{l,i}$ and points $y_{\alpha,l,i}$ for $\alpha < \omega_1$, $1 \leqslant l \leqslant n$ and $1 \leqslant i \leqslant s_l$ for some $s_l \in N$ such that

$$\nu_\alpha^l = \sum_{1\leqslant i\leqslant s_l} q_{l,i}\delta_{y_{\alpha,l,i}}.$$

Using the fact that for $k = \Sigma_{1\leqslant k\leqslant n} s_l$ the space K^k is hereditarily separable and $h_{\alpha,l,j}$'s are continuous functions we find $\alpha < \beta < \omega_1$ such that

$$\left|h_{\beta,l,j}(y_{\alpha,l,i}) - h_{\beta,l,j}(y_{\beta,l,i})\right| < \varepsilon/3kL$$

where $L > |q_{l,j}|$ for all $1 \leqslant l \leqslant n$, $1 \leqslant j \leqslant p_l$ and $1 \leqslant i \leqslant s_l$. So,

$$\begin{aligned}\left|\mu_\alpha^l(h_{\beta,l,j}) - \mu_\beta^l(h_{\beta,l,j})\right| &\leqslant \left|\nu_\alpha^l(h_{\beta,l,j}) - \nu_\beta^l(h_{\beta,l,j})\right| + \left\|\nu_\alpha^l - \mu_\alpha^l\right\|M + \left\|\nu_\beta^l - \mu_\beta^l\right\|M \\ &\leqslant \sum_{1\leqslant i\leqslant k} |q_{l,j}|\left|h_{\beta,l,j}(y_{\alpha,l,i}) - h_{\beta,l,j}(y_{\beta,l,i})\right| + 2\varepsilon/3 \leqslant \varepsilon\end{aligned}$$

showing that x_α^* belongs to the neighbourhood U_β of x_β^* that is showing that $(x_\alpha^*)_{\alpha<\omega_1}$ is not left-separated and completing the proof of the fact that every finite power of $l_1(K)$ is hereditarily separable in the weak* topology and by Proposition 6 that every finite power of $M(K)$ is hereditarily separable.

Now it remains to prove that $C(K)$ is hereditarily Lindelöf in the weak topology. Consider the following notation for basic sets in the weak topology on $C(K)$ and the basic sets in the weak* topology on $C^*(K)$ respectively:

$$V(\mu_1,\dots,\mu_n; I_1,\dots,I_n) = \left\{f \in C(K)\colon \forall 1 \leqslant i \leqslant n,\ \mu_i(f) \in I_i\right\},$$
$$V(f_1,\dots,f_n; I_1,\dots,I_n) = \left\{\mu \in M(K)\colon \forall 1 \leqslant i \leqslant n,\ \mu(f_i) \in I_i\right\}.$$

Suppose that $C(K)$ is not hereditarily Lindelöf in the weak topology, this w.l.o.g. implies that there are $f_\alpha \in C(K)$ and $\mu_\alpha^1,\dots,\mu_\alpha^n \in M(K)$ such that $f_\alpha \in V(\mu_\alpha^1,\dots,\mu_\alpha^n; I_1,\dots,I_n)$ and $f_\beta \notin V(\mu_\alpha^1,\dots,\mu_\alpha^n; I_1,\dots,I_n)$ for $\beta > \alpha$. However note that

$$f \in V(\mu_1,\dots,\mu_n; I_1,\dots,I_n) \quad \text{iff} \quad (\mu_1,\dots,\mu_n) \in V(f, I_1) \times \cdots \times V(f, I_n).$$

That is $(\mu_\alpha^1,\dots,\mu_\alpha^n) \in V(f_\alpha, I_1) \times \cdots \times V(f_\alpha, I_n)$ and $(\mu_\alpha^1,\dots,\mu_\alpha^n) \notin V(f_\beta, I_1) \times \cdots \times V(f_\beta, I_n)$ that is $(\mu_\alpha^1,\dots,\mu_\alpha^n)$s form a left-separated sequence in $M(K)^n$ which contradicts the fact that this space is hereditarily separable. □

Remark. Recall some limitations of the above result: Katetov's theorem says that if a compact K is nonmetrizable, then K^3 is not hereditarily Lindelöf and it follows from Todorcevic's results in [17] that one cannot prove in ZFC the existence of a nonseparable $C(K)$ which is hereditarily Lindelöf in the weak topology as biorthogonal sequences form discrete subspaces in such a topology.

3. The existence of $\mathcal{A}$

In this section we prove Theorem 7. The algebra $\mathcal{A}$ will be constructed as the direct limit (i.e., corresponding to the union of algebras) of a directed system of finite subalgebras with distinguished generators satisfying finite versions of the requirements from Lemma 1. To express them properly we will need some terminology and a few definitions. By $At(A)$ we will mean the collection of all atoms of a Boolean algebra A. $B = A[a_1, \dots, a_n]$ will mean that the algebra B is generated over A by its elements $a_1, \dots, a_n$.

Definition 9. Suppose $A[a]$ is a finite Boolean algebra. We say that $a \notin A$ is minimal over A if and only if there is only one atom of A which is not an atom of $A[a]$.

It is clear that if a is minimal over A, then the only atom mentioned in the definition above is the supremum of two atoms of $A[a]$. Such an atom will be called a *split* atom. For a general notion of a minimal extension see [5] or [6] in the forcing context.

Definition 10. Let $B = A[a_1, \dots, a_n]$ and let $x_1, \dots, x_n$ be distinct atoms of A. We say that $a_1, \dots, a_n$ are strong minimal generators of B over A splitting $x_1, \dots, x_n$ if and only if for every $1 \leqslant i \leqslant n$ the element a_i is minimal over $A[a_1, \dots, a_{i-1}]$ and the only atom of $A[a_1, \dots, a_{i-1}]$ split by a_i is also an atom of A equal to x_i.

Note that a_2 may not be minimal over A, since it may split x_1, but it does not split the two atoms of $A[a_1]$ which are below x_1. We will need two simple lemmas about the above notions which we leave without proofs.

Lemma 11. *Let A be a finite Boolean algebra. Suppose that $x_0 \in At(A)$ and $x, y \in A$ are such that $x \wedge x_0 = y \wedge x_0 = x \wedge y = 0_A$, $x \vee y \vee x_0 = 1_A$. Then there is a unique (up to isomorphism preserving A) one-extension $A[a]$ such that a is minimal over A and the following relations hold in $A[a]$:*

$$x \leqslant a,$$
$$y \wedge a = 0.$$

In this case x_0 is the only atom of A split in $A[a]$.

Lemma 12. *Suppose that A and B are finite Boolean algebras and $i : A \to B$ is a monomorphism of Boolean algebras (including $i(1_A) = 1_B$). Suppose that $x_0 \in At(A)$, x, y are as in the previous lemma, a is minimal over A and satisfies*

$$x \leqslant a,$$
$$y \wedge a = 0.$$

Suppose that $b \in B - i[A]$ satisfies

$$i(x) \leqslant b,$$
$$i(y) \wedge b = 0.$$

Then there is a monomorphism $j : A[a] \to B$ of Boolean algebras such that $i \subseteq j$ and $j(a) = b$.

Proof. It follows from the previous lemma. □

Now we are ready to define our partial order of finite approximations. Fix an uncountable sequence $(r_\xi\colon \xi < \omega_1)$ of distinct elements of $\{-1, 1\}^\omega$ and an independent family $(U_i\colon i \in N)$.

We define P to be the partial order whose elements are of the form

$$p = \bigl(A_p, F_p, n_p, \bigl(a^p_\alpha\colon \alpha \in F_p\bigr)\bigr)$$

where:

(1) $F_p \subseteq \omega_1$ is finite,
(2) n_p is a positive integer such that $r_\alpha|n_p \neq r_\beta|n_p$ for distinct $\alpha, \beta \in F_p$,
(3) $a^p_{\alpha_1}, \dots, a^p_{\alpha_k}$ are strong minimal generators of A_p over the subalgebra generated by $(U_i\colon i < n_p)$ splitting $U_{r_{\alpha_1}|n_p}, \dots, U_{r_{\alpha_k}|n_p}$, where $\{\alpha_1, \dots, \alpha_k\}$ is an increasing enumeration of F_p.

For $p, q \in P$, we say that $p \leqslant q$ if and only if

(4) $n_p \geqslant n_q$, $F_p \supseteq F_q$ and,
(5) there is a monomorphism of Boolean algebras $\phi : A_q \to A_p$ such that $\phi(U_i) = U_i$ for $i < n_q$ and $\phi(a^q_\alpha) = a^p_\alpha$ for all $\alpha \in F_q$.

Note that with the above notation we have that $U_{r_\beta|n_p} \wedge \delta a_\beta \leqslant a_\alpha$ or $U_{r_\beta|n_p} \wedge \delta a_\beta \leqslant -a_\alpha$ for every $\beta < \alpha$ from F_p and $\delta \in \{-1, 1\}$ and that $U_s \leqslant a_\alpha$ or $U_s \leqslant -a_\alpha$ for $s \in \{-1, 1\}^{n_p} - \{r_\beta|n_p\colon \beta \in F_p \cap \alpha\}$ i.e., we have conditions corresponding to requirements (a)–(c) from Definition 1, and the corresponding forms of the ultrafilters. It follows, in particular, that the atoms of A_p are of the form $U_{r_\alpha|n_p} \wedge a^p_\alpha$, $U_{r_\alpha|n_p} - a^p_\alpha$ for $\alpha \in F_p$ and U_s for $s \in \{-1, 1\}^{n_p} - \{r_\alpha|n_p\colon \alpha \in F_p\}$.

Now we prove a lemma which provides two ways of amalgamating the finite approximations (to our uncountable algebra) from P.

Lemma 13. *Suppose that p and q are two conditions of P such that $n_p = n_q = n$, $F_p = \{\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_{m+k}\}$, $F_q = \{\alpha_1, \dots, \alpha_m, \alpha_{m+k+1}, \dots, \alpha_{m+2k}\}$, with $\alpha_i < \alpha_j$ for integers $0 < i < j \leqslant m + 2k$. Suppose that $r_{\alpha_i}|n = r_{\alpha_{k+i}}|n$ for $m < i \leqslant k$ and that there is an isomorphism of Boolean algebras $\psi : A_p \to A_q$ such that $\psi(U_i) = U_i$, $\psi(a^p_{\alpha_i}) = a^q_{\alpha_i}$ for $1 \leqslant i \leqslant m$ and $\psi(a^p_{\alpha_i}) = a^q_{\alpha_{k+i}}$ for $m < i \leqslant k$. Let $\epsilon_i \in \{-1, 1\}$, $\delta_i \in \{-1, 1\}$ for $1 \leqslant i \leqslant k$ and let $a \subseteq \{m + 1, \dots, m + k\}$.*

Then there are conditions $t, v \leqslant p, q$ of P such that:

(a) $U_{r_{\alpha_{k+i}}|n_t} \leqslant a^t_{\alpha_i}$, *if* $i \in a$ *and* $U_{r_{\alpha_{k+i}}|n_t} \leqslant -a^t_{\alpha_i}$, *if* $i \notin a$,
(b) $U_{r_{\alpha_i}|n_v} \wedge a^v_{\alpha_i} \leqslant \epsilon_i a^v_{\alpha_{k+i}}$, $U_{r_{\alpha_i}|n_v} - a^v_{\alpha_i} \leqslant \delta_i a^v_{\alpha_{k+i}}$ *for all* $m < i \leqslant k$.

Proof. First let us define t. Put $F_t = F_p \cup F_q$. Take n_t to be sufficiently large so that (2) of the definition of P is satisfied. Let A_t be the Boolean algebra generated by $(U_i\colon i < n_t)$ and elements x_α for $\alpha \in F_t$ which satisfy

$$0 < x_\alpha < U_{r_\alpha|n_t}.$$

For $\alpha \in F_t$ let $r(\alpha)$ be p if $\alpha \in F_p$ and q otherwise. We need to define the generators $a^t_{\alpha_i}$ for $1 \leqslant i \leqslant m + 2k$ of the above algebra A_t. We do it by induction on $1 \leqslant i \leqslant m + 2k$ putting $a^t_{\alpha_i} = c_{\alpha_i} \vee d_{\alpha_i}$ for $i \in a$ and $a^t_{\alpha_i} = c_{\alpha_i} \vee x_{\alpha_i}$ for $i \notin a$ where

$$\begin{aligned} c_{\alpha_i} = &\bigvee \bigl\{U_s\colon U_s \leqslant a^{r(\alpha_i)}_{\alpha_i},\ s \in \{-1, 1\}^n,\ s \neq r_{\alpha_i}|n\bigr\} \\ &\vee \bigvee \bigl\{U_{r_{\alpha'}|n} \wedge a^t_{\alpha'}\colon U_{r_{\alpha'}|n} \wedge a^{r(\alpha_i)}_{\alpha'} \leqslant a^{r(\alpha_i)}_{\alpha_i},\ \alpha' \in \alpha_i \cap F_{r(\alpha_i)}\bigr\} \\ &\vee \bigvee \bigl\{U_{r_{\alpha'}|n} - a^t_{\alpha'}\colon U_{r_{\alpha'}|n} - a^{r(\alpha_i)}_{\alpha'} \leqslant a^{r(\alpha_i)}_{\alpha_i},\ \alpha' \in \alpha_i \cap F_{r(\alpha_i)}\bigr\}, \end{aligned}$$

$$d_{\alpha_i} = [U_{r_{\alpha_i}|n} - U_{r_{\alpha_i}|n^t}] \vee x_{\alpha_i}.$$

First note that the above definition implies that

$$a^t_{\alpha_1}, \ldots, a^t_{\alpha_m}, a^t_{\alpha_{m+1}}, \ldots, a^t_{\alpha_m+k}, a^t_{\alpha_m+k+1}, \ldots, a^t_{\alpha_{m+2k}}$$

are strong minimal generators of A_t over the subalgebra generated by $(U_i\colon i < n_t)$ splitting $U_{r_{\alpha_1}|n_t}, \ldots, U_{r_{\alpha_{m+2k}}|n_t}$. That is t is a condition of P.

To prove that $t \leqslant p, q$ it is enough to construct the monomorphisms $\phi_p : A_p \to A_t$ and $\phi_t : A_q \to A_t$ required in (5). We start with the identity on the subalgebra generated by $(U_i\colon i < n)$ and keep extending it using Lemma 12 obtaining at the end the required monomorphisms. The hypothesis of Lemma 12 is satisfied at each stage by the form of the definition of c_α and the fact that p and q are isomorphic in the sense of the hypothesis on ψ from Lemma 13.

Finally to prove (a) of Lemma 13 we note that for $m < i \leqslant k$ we have

$$U_{r_{\alpha_{k+i}}|n_t} \wedge U_{r_{\alpha_i}|n_t} = 0,$$

$$U_{r_{\alpha_{k+i}}|n_t} \subseteq U_{r_{\alpha_{k+i}}|n},$$

$$c_{\alpha_i} \wedge U_{r_{\alpha_{k+i}}|n} = 0,$$

so by the definition of d_α and x_α we conclude (a).

The idea of the construction of v is similar, define n_v, F_v, x_α and A_v and c_{α_i} as above. Now $a^t_{\alpha_i} = c_{\alpha_i} \vee x_{\alpha_i}$ for $1 \leqslant i \leqslant m + k$. For $m + k < i \leqslant m + 2k$ we define $a^t_{\alpha_i} = c_{\alpha_i} \vee d_{\alpha_i} \vee e_{\alpha_i}$ where $d_{\alpha_i} = U_{r_{\alpha_{i-k}}|n_v} \wedge a^v_{\alpha_i}$ if $\epsilon_i = 1$ and $d_{\alpha_i} = 0$ otherwise, and $e_{\alpha_i} = U_{r_{\alpha_{i-k}}|n_v} - a^v_{\alpha_i}$ if $\delta_i = 1$ and $e_{\alpha_i} = 0$ otherwise. The same argument as before shows that v is a condition of P satisfying (b) and $v \leqslant p, q$. $\square$

Remark. It is clear from the definition of the finite approximations in P that the direct limit of any directed system of elements of P will be like in Definition 1. Hence we are facing the problem of choosing the right directed system. Some exist without the necessity of making any additional set-theoretic assumptions. However in order to prove Theorem 7, we need the direct limit to satisfy a property of the sort *among any uncountably many finite subalgebras of $\mathcal{A}$ there are two which are amalgamated like in Lemma* 13. $\diamondsuit$ provides this possibility and [15] attempts an axiomatized approach to these constructions which is quite complex however. Here we present another technically simple approach using forcing (see [7]).

Proof of Theorem 7. We will show that in the generic extension obtained by forcing with P (see e.g., [7, Chapter 7]) there is a Boolean algebra satisfying the statement of Theorem 7. Of course the Boolean algebra $\mathcal{A}$ is the direct limit of the algebras A_p for $p \in G$, where G is a generic filter in P. The possibility of the amalgamations as in Lemma 13 implies that $\mathcal{A}$ is uncountable. Suppose that we are given P-names for pairwise disjoint sets $\dot{F}_\alpha = \{\dot{\xi}_\alpha^1, \ldots, \dot{\xi}_\alpha^k\}$ for $\alpha < \omega_1$ and P-names $\dot{\epsilon}^i$ and $\dot{\delta}^i$ for elements of $\{-1, 1\}$ for all $1 \leqslant i \leqslant k$ and $\dot{a} \subseteq \{1, \ldots, k\}$. Choose conditions $p_\alpha \in P$ and sets $F_\alpha = \{\xi_\alpha^1, \ldots, \xi_\alpha^k\}$ for $\alpha < \omega_1$ and numbers $\epsilon^i \in \{-1, 1\}$, $\delta^i \in \{-1, 1\}$ for all $1 \leqslant i \leqslant k$ and $a \subseteq \{1, \ldots, k\}$ such that p_α forces that $\check{F}_\alpha = \dot{F}_\alpha$, $\check{\xi}_\alpha = \dot{\xi}_\alpha$, $\check{\epsilon}^i = \dot{\epsilon}^i$ and $\check{\delta}^i = \dot{\delta}^i$ and $\check{a} = \dot{a}$. By choosing a Δ-system of sets and extending the conditions we may assume without loss of generality that the sets F_α are pairwise disjoint and $F_\alpha \subseteq F_{p_\alpha}$. We may find two p_α, p_β which satisfy the hypothesis of Lemma 13. By this lemma there are two amalgamations of p_α and p_β. The first forces the first part of Theorem 7 and the second amalgamation forces the second part of Theorem 7. This argument also shows that P satisfies the c.c.c., hence it preserves ω_1, i.e., $\mathcal{A}$ is uncountable in the generic extension. $\square$

4. Weak* right-separated sequences in the dual space

Recall the definitions of right-separated sequences from the introduction and the fact that dual balls of nonseparable Banach spaces always have uncountable right-separated sequence obtained from a sequence $(f_\alpha, \mu_\alpha)_{\alpha<\omega_1} \subseteq C(K) \times C(K)^*$ such that $\mu_\alpha(f_\alpha) = 1$ and $\mu_\beta(f_\alpha) = 0$ for all $\alpha < \beta < \omega_1$. The following lemma (and its claims) shows that the only way of constructing such a sequence in our $C(K)$ is to chose μ_α close to $\pm(\delta_{[r_\xi,1]} - \delta_{[r_\xi,-1]})$ and f_α essentially χ_{a_ξ} or χ_{-a_ξ}, which in the case of the algebra $\mathcal{A}$ i.e., our $C(K)$ gives $\xi < \eta$ such that $\mu_\xi(f_\eta) = -1$.

Lemma 14. *Suppose that $(f_\alpha)_{\alpha<\omega_1}$ is a sequence of rational simple functions on K and $(\mu_\alpha)_{\alpha<\omega_1}$ a sequence of Radon measures on K. Then either there are $\alpha < \beta$ such that*

$$\left|\int f_\alpha \, d\mu_\beta\right| > 0.01 \tag{a}$$

or there is $\alpha \in \omega_1$ such that

$$\int f_\alpha \, d\mu_\alpha < 0.99 \tag{b}$$

or there are $\alpha < \beta < \omega_1$ such that

$$\int f_\beta \, d\mu_\alpha < -0.85. \tag{c}$$

Proof. By the separability of $C(\{-1, 1\}^\omega)$, Lemma 4 and thinning out the sequence, for all $\alpha < \omega_1$ we may assume to have

$$f_\alpha = g + \sum_{1 \leqslant i \leqslant k} q_i \chi_{a^*_{\xi^i_\alpha} \cap U^*_{s_i}}$$

for some simple rational function $g \in C(\{-1,1\}^\omega)$ and $F_\alpha = \{\xi_\alpha^1, \dots, \xi_\alpha^k\} \subseteq \omega_1$ and some $s_i \in 2^{m_i}$, $m_i \in N$ and for some rationals q_i and $1 \leqslant i \leqslant k$ such that $s_i = r_{\xi_\alpha^i}|m_i$ and such that

$$\sum_{1\leqslant i\leqslant k} |q_i||\mu_\alpha|\big(U_{s_i}^* - R_{\xi_\alpha^i}\big) \leqslant 0.01.$$

By thinning out the sequence (applying the Δ-system lemma, see [7]) and moving some identical parts to g we may assume that F_α's are pairwise disjoint and g (no longer in $C(\{-1,1\}^\omega)$) is fixed.

Claim 1. *Either* (a) *holds or for all but countably many α's in ω_1 we have*

$$\left|\int g\,d\mu_\alpha\right| \leqslant 0.02.$$

Proof. If $|\int g\,d\mu_\alpha| > 0.02$ for uncountably many ordinals, by Theorem 7 we can find among them $\alpha < \beta < \omega_1$ such that $R_{\xi_\beta^i} \subseteq -a^*_{\xi_\alpha^i}$ for all $1 \leqslant i \leqslant k$. This means that

$$\begin{aligned}\left|\int f_\alpha\,d\mu_\beta\right| &\geqslant \left|\int g\,d\mu_\beta\right| - \sum|q_i|\left|\int \chi_{a^*_{\xi_\alpha^i}\cap U^*_{s_i}}\,d\mu_\beta\right|\\ &\geqslant \left|\int g\,d\mu_\beta\right| - \sum|q_i||\mu_\beta|\big(U_{s_i}^* - R_{\xi_\beta^i}\big) > 0.02 - 0.01 = 0.01\end{aligned}$$

obtaining (a) of the lemma. □

Claim 2. *Either* (a) *holds or for uncountably many α's in ω_1 we have*

$$\sum_{1\leqslant i\leqslant k} \big|q_i\mu_\alpha(R_{\xi_\alpha^i})\big| \leqslant 0.08.$$

Proof. Suppose that the condition above does not hold. Without loss of generality we may assume that the condition from Claim 1 holds for all $\alpha < \beta < \omega_1$ and that there is a fixed (for all $\alpha < \omega_1$) subset $a \subseteq \{1,\dots,k\}$ such that $i \in a$ if and only if $q_i\mu_\alpha(R_{\xi_\alpha^i})$ is non-negative. So we have

$$\left|\sum_{i\in a} q_i\mu_\alpha(R_{\xi_\alpha^i})\right| > 0.08/2 \quad \text{or} \quad \left|\sum_{i\notin a} q_i\mu_\alpha(R_{\xi_\alpha^i})\right| > 0.08/2.$$

Let us assume the first case, the second is analogous. Using Theorem 7 we obtain $\alpha < \beta$ such that

$$\begin{aligned} R_{\xi_\beta^i} &\subseteq -a^*_{\xi_\alpha^i}, \quad \text{if } i \notin a,\\ R_{\xi_\beta^i} &\subseteq a^*_{\xi_\alpha^i}, \quad \text{if } i \in a.\end{aligned}$$

So we obtain

$$\begin{aligned}\left|\int f_\alpha\, d\mu_\beta\right| &\geqslant \left|\sum_{i\in a} q_i \int \chi_{a^*_{\xi^i_\alpha}\cap U^*_{s_i}}\, d\mu_\beta\right| - \left|\int g\, d\mu_\beta\right| - \sum_{i\notin a} |q_i||\mu_\beta|\big(U^*_{s_i} - R_{\xi^i_\beta}\big)\\ &\geqslant \left|\sum_{i\in a} q_i \mu_\beta(R_{\xi^i_\beta})\right| - \left|\int g\, d\mu_\beta\right| - \sum_{1\leqslant i\leqslant k} |q_i||\mu_\beta|\big(U^*_{s_i} - R_{\xi^i_\beta}\big)\\ &> 0.08/2 - 0.02 - 0.01 = 0.01\end{aligned}$$

obtaining (a). □

Claim 3. *Either* (a) *or* (b) *holds or for uncountably many α's in ω_1 we have*

$$\sum_{1\leqslant i\leqslant k} q_i \mu_\alpha\big([r_{\xi^i_\alpha}, -1]\big) \leqslant -0.88.$$

Proof. Suppose that (a) and (b) does not hold, i.e., the conditions of Claims 1 and 2 hold and that the condition above does not hold. We will obtain a contradiction.

By applying Claims 1 and 2 and the fact that

$$R_{\xi^i_\alpha} = \big\{[r_{\xi^i_\alpha}, 1], [r_{\xi^i_\alpha}, -1]\big\}$$

as well as the fact that

$$\chi_{a^*_{\xi^i_\alpha}\cap U^*_{s_i}}\big([r_{\xi^i_\alpha}, 1]\big) = 1,$$
$$\chi_{a^*_{\xi^i_\alpha}\cap U^*_{s_i}}\big([r_{\xi^i_\alpha}, -1]\big) = 0,$$

from the negation of (b) we obtain that

$$\sum_{1\leqslant i\leqslant k} q_i \mu_\alpha\big(\big\{[r_{\xi^i_\alpha}, 1]\big\}\big) \geqslant 0.99 - 0.02 - 0.01 = 0.96.$$

So, if

$$\sum_{1\leqslant i\leqslant k} q_i \mu_\alpha\big([r_{\xi^i_\alpha}, -1]\big) > -0.88,$$

we get that

$$\sum_{1\leqslant i\leqslant k} q_i \mu_\alpha(R_{\xi^i_\alpha}) > 0.08$$

which contradicts Claim 2 and completes the proof of Claim 3. □

To finish the proof of the lemma, we need to assume that (a) and (b) fail, i.e., all the conditions of the claims hold, and we need to get (c). By Theorem 7 we can find $\alpha < \beta < \omega_1$ such that

$$[r_{\xi^i_\alpha}, -1] \in a^*_{\xi^i_\beta},$$
$$[r_{\xi^i_\alpha}, 1] \notin a^*_{\xi^i_\beta}$$

for all $1 \leqslant i \leqslant k$. This implies that

$$\begin{aligned}\int f_\beta \, d\mu_\alpha &= \sum_{1\leqslant i\leqslant k} \int_{U^*_{s_i}} q_i \chi_{a^*_{\xi^i_\beta}} \, d\mu_\alpha + \int g \, d\mu_\alpha \\ &\leqslant \sum_{1\leqslant i\leqslant k} q_i \mu_\alpha\big([r_{\xi^i_\alpha}, -1]\big) + \sum_{1\leqslant i\leqslant k} |\mu_\alpha|\big(U^*_{s_i} - R_{\xi^i_\alpha}\big) + \left|\int g \, d\mu_\alpha\right| \\ &\leqslant -0.88 + 0.01 + 0.02 = -0.85\end{aligned}$$

which completes the proof of the lemma. □

Theorem 15. *There are no uncountable semiorthogonal sequences in* $(C(K), C(K)^*)$ *where* K *is the Stone space of the Boolean algebra* $\mathcal{A}$. *There is no support set in* $C(K)$.

Proof. We use the characterization of spaces with the support set from [2] (see the introduction) as those which have no uncountable semibiorthogonal sequences. Suppose $(f_\alpha, \mu_\alpha)_{\alpha<\omega_1} \subseteq C(K) \times C(K)^*$. We may assume without loss of generality that $\|\mu_\alpha\| \leqslant M$ for some positive M. By the Stone–Weierstrass theorem we can chose $f'_\alpha \in C(K)$ which is a rational simple function and

$$\|f'_\alpha - f_\alpha\| < 0.01/M.$$

This means that (a) and (b) of Lemma 14 do not hold, for f'_α's instead of f_α's i.e., (c) holds which implies that $(f_\alpha, \mu_\alpha)_{\alpha<\omega_1}$ is not semibiorthogonal. □

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